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EVOLUTION OF A REGULAR PRECESSION OF A SOLID BODY CARRYING A VISCO-ELASTIC DISK*

V.G. VIL'KE

The motion of inertia is studied of a system consisting of an axisymmetric solid body with fixed point and a homogeneous visco-elastic disk lying in the equatorial plane of the ellipsoid of inertia of the solid body (the center of disk coincides with the fixed point). In the case of a solid disk immobilized relative to the solid body the system accomplishes a regular precession (the case of Euler motion of a symmetric solid body with a fixed point /1/). The deformation of the disk is taking place in the plane of the disk, and is accompanied by energy dissipation is the cause of the regular precession finishing by steady rotation about the vector of the moment of momentum of the system /2/.

For defining the body position we use the Euler angles /1/. We introduce three systems of coordinates: system $O\xi_1\xi_2\xi_3$ is fixed, system $Ox_1x_2x_3$ is attached to the solid body (axis Ox_3 is the axis of symmetry of the body), and the system $Oxyz$ is obtained from the fixed system by two rotations by the Euler angles ψ and θ. The disk is located in plane Ox_1x_2 and displacements of its points, which are small, occurs in a plane Ox_1x_2, and the stresses correspond to the plane stress state /3/.

The equations of motion of the system are of the form /2/

$$J[\mathbf{u}]\boldsymbol{\omega}^{\cdot}+\boldsymbol{\omega}\times J[\mathbf{u}]\boldsymbol{\omega}+\int_{\Omega}(\mathbf{r}+\mathbf{u})\times[\mathbf{u}^{\cdot\cdot}+2\boldsymbol{\omega}\times\mathbf{u}^{\cdot}]\rho\,dx=0 \tag{1}$$

$$\int_{\Omega}\{\boldsymbol{\omega}^{\cdot}\times(\mathbf{r}+\mathbf{u})+\boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\mathbf{r}+\mathbf{u})]+\mathbf{u}^{\cdot\cdot}+2\boldsymbol{\omega}\times\mathbf{u}^{\cdot}\}\,\delta\mathbf{u}\rho\,dx+$$
$$(\nabla\mathrm{E}[\mathbf{u}]+\nabla D[\mathbf{u}^{\cdot}],\delta\mathbf{u})=0\quad\forall\delta\mathbf{u}\in V,\quad dx=dx_1\,dx_2$$

where $\mathbf{u}(\mathbf{r},t)$ is the vector of displacement of elastic disk points, $J[\mathbf{u}]$ is the tensor of inertia of the system relative to the axes $Ox_1x_2x_3$, $\boldsymbol{\omega}$ is the angular velocity of rotation of the solid body, Ω is the region taken by the disk in the natural state, $\mathrm{E}[\mathbf{u}], D[\mathbf{u}^{\cdot}]$ are the functionals of potential energy of elastic deformation and dissipative forces, and ρ is the mass of unit area of the plate. Configuration of the system space is $SO(3)\times V$, where $SO(3)$ is the group of rotation of three-dimensional space, and $V=\{\mathbf{u}:\mathbf{u}\cdot(W_2^1(\Omega))^2,\ \mathbf{u}(0,t)=0\}$, $W^1(\Omega)$ is the Sobolev space.

From Eqs.(1) follows the law of conservation of moment of momentum

$$\mathbf{G}=J[\mathbf{u}]\boldsymbol{\omega}+\int_{\Omega}[(\mathbf{r}+\mathbf{u})\times\mathbf{u}^{\cdot}]\rho\,dx \tag{2}$$

where vector $\mathbf{G}$ is continuous and directed along axis $O\xi_3$.

Let us obtain the approximate equations defining the evolution of regular precession of the solid body. We take the regular precession as the unperturbed motion when defining the disk deformation from the second of Eqs.(1). In that case $[\boldsymbol{\omega}^{\cdot}\times(\mathbf{r}+\mathbf{u})]\,\delta\mathbf{u}=0$, since the vectors appearing in the mixed product lie in the plane Ox_1x_2.

Suppose the conditions are valid /4,5/

$$|\mathbf{u}|\ll|\mathbf{r}|,\quad|\boldsymbol{\omega}|\ll\nu_1,\quad D[\mathbf{u}]=\chi\mathrm{E}[\mathbf{u}] \tag{3}$$

where ν_1 is the natural frequency of oscillation of the disk on a lower harmonic, and χ is a constant. Conditions (3) allow us to neglect terms $\mathbf{u}^{\cdot\cdot}$ and $2\boldsymbol{\omega}\times\mathbf{u}^{\cdot}$ (the relative and the Coriolis accelerations in system $Ox_1x_2x_3$) in the second of Eqs.(1) and the problem of disk deformation may be regarded as a problem of quasi-statics

$$(\nabla D[\mathbf{u}^{\cdot}]+\nabla\mathrm{E}[\mathbf{u}],\delta\mathbf{u})=-\int_{\Omega}[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})]\,\delta\mathbf{u}\rho\,dx\quad\forall\delta\mathbf{u}\in V \tag{4}$$

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$$\mathbf{u}(\mathbf{r},t)=\sum_{k=0}^{\infty}(-\chi)^k\frac{\partial^k \mathbf{u}_0(\mathbf{r},t)}{\partial t^k} \tag{5}$$

where $\mathbf{u}_0(\mathbf{r},t)$ is the solution of the variational equation of the plane problem of the theory of elasticity

$$(\nabla \mathbf{E}[\mathbf{u}_0],\delta\mathbf{u})=-\int_{\Omega}[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})]\,\delta\mathbf{u}\rho\,dx \quad \forall\delta\mathbf{u}\in V \tag{6}$$

and satisfies the boundary conditions

$$\mathbf{u}_0(0,t)=0,\quad \sigma_n|_{\Gamma}=0,\quad \Gamma=\partial\Omega=\{(x_1,x_2):x_1^2+x_2^2=l^2\} \tag{7}$$

The convergence of series (5) depends on $\chi|\varphi^{\cdot}|$. After centrifugal forces have been eliminated in the right-hand side of Eq.(6) it becomes stationary in the coordinate system $Oxyz$, it assumes the form /3/

$$\begin{aligned}
&\operatorname{grad}\operatorname{div}\mathbf{u}_0'+\mu_1\Delta\mathbf{u}_0'=\mu_2P_3[\boldsymbol{\omega}'\times(\boldsymbol{\omega}'\times\mathbf{r}')]\\
&\mu_1=(1-\sigma)(1+\sigma)^{-1},\quad \mu_2=2\rho(1-\sigma)(Eh)^{-1},\ \mathbf{r}'(x_1',x_2',0),\ \boldsymbol{\omega}'(0,Q,R+\varphi^{\cdot})\\
&Q=\psi^{\cdot}\sin\theta,\quad R=\psi^{\cdot}\cos\theta,\quad P_3[\boldsymbol{\omega}'\times(\boldsymbol{\omega}'\times\mathbf{r}')]=-[Q^2+(R+\\
&\quad\varphi^{\cdot})^2]x_1'\mathbf{e}_x-(R+\varphi^{\cdot})^2x_2'\mathbf{e}_y
\end{aligned} \tag{8}$$

where the prime denotes vectors relative to the system of coordinates $Oxyz$, P_3 is the operator of projection on the plane Oxy, h is the disk thickness, E and σ are, respectively, the module of elasticity and the coefficient of Poisson of disk material, $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit vectors of axes Ox and Oy. We obtain the solution of Eq.(8) in the form /5/

$$\mathbf{u}_0'=(a_{11}x_1'^2+a_{12}x_2'^2+c_1)x_1'\mathbf{e}_x+(a_{21}x_1'^2+a_{22}x_1'^2+c_2)x_2'\mathbf{e}_y \tag{9}$$

Substituting (9) into (8) and (7), after transformation, we obtain the system of six linear algebraic equtions with constant coefficients $a_{11}, a_{12}, a_{21}, a_{22}, c_1, c_2$, whose solution is

$$\begin{aligned}
&\mathbf{u}_0'(\mathbf{r}')=g[\mathbf{r}'^2-l^2(2+\mu_1)]\mathbf{r}'+D\mathbf{r}',\quad D=\operatorname{diag}\{d_1,d_2,0\}\\
&d_i=(B_i\mathbf{r}',\mathbf{r}')+g_i,\quad i=1,2\\
&B_1=k_1\operatorname{diag}\{2+5\mu_1,\ 6+9\mu_1,\ 0\},\quad g_1=k_2(\mu_1^2+4\mu_1+2)\\
&B_2=k_1\operatorname{diag}\{-6-3\mu_1,\ -2+\mu_1,\ 0\},\ g_2=k_2(\mu_1^2-2)\\
&g=-\frac{\mu_2(R+\varphi')^2}{8(1+\mu_1)},\quad k_1=-\frac{\mu_2Q^2}{48\mu_1(1+\mu_1)},\quad k_2=\frac{l^2\mu_2Q^2}{16\mu_1(1+\mu_1)}
\end{aligned}$$

Since $\mathbf{r}'=A(t)\mathbf{r}$, where $A(t)$ is the matrix for passing from the system of coordinates $Ox_1x_2x_3$ to $Qxyz$, hence

$$\begin{aligned}
&\mathbf{u}_0(\mathbf{r},t)=A^{-1}\operatorname{diag}\{(A^{-1}B_1A\mathbf{r},\ \mathbf{r})+g_1,\ (A^{-1}B_2A\mathbf{r},\mathbf{r})+g_2,0\}A\mathbf{r}+\\
&\quad g[\mathbf{r}^2-l^2(2+\mu_1)]\mathbf{r},\ A=(\gamma_{ij}),\ \gamma_{11}=\gamma_{22}=\cos\varphi\\
&\gamma_{21}=-\gamma_{12}=\sin\varphi,\ \gamma_{33}=1,\ \gamma_{13}=\gamma_{31}=\gamma_{32}=\gamma_{23}=0
\end{aligned} \tag{10}$$

The series (5) is convergent when $4\chi|\varphi^{\cdot}|<1$. Taking $4\chi|\varphi^{\cdot}|$ as fairly small, we restrict subsequently (5) to two first terms, assuming

$$\mathbf{u}(\mathbf{r},t)=\mathbf{u}_0(\mathbf{r},t)-\chi\mathbf{u}_0^{\cdot}(\mathbf{r},t) \tag{11}$$

Note that according to (10) and (11) function $\mathbf{u}(\mathbf{r},t)$ is of the form

$$\begin{aligned}
&\mathbf{u}(\mathbf{r},t)=\mathbf{v}_0(\mathbf{r})+\mathbf{v}_1(\mathbf{r})(\cos2\varphi+2\chi\varphi^{\cdot}\sin2\varphi)+\mathbf{v}_2(\mathbf{r})(\sin2\varphi-\\
&\quad 2\chi\varphi^{\cdot}\cos2\varphi)+\mathbf{w}_1(\mathbf{r})(\cos4\varphi+4\chi\varphi^{\cdot}\sin4\varphi)+\mathbf{w}_2(\mathbf{r})(\sin4\varphi-4\chi\varphi^{\cdot}\cos4\varphi)
\end{aligned} \tag{12}$$

We substitute the displacement (12) in the first of Eqs.(1) and to integral (2), we average the obtained equation over the "rapid" time (over the angle φ) and find the approximate equations defining the evolution of the evolution of the "slow" variables θ and $\psi^{\cdot}$. From the first of Eqs.(1) we have

$$\begin{aligned}
&J_{11}[\mathbf{u}]p^{\cdot}-J_{12}[\mathbf{u}]q^{\cdot}+(J_{33}[\mathbf{u}]-J_{22}[\mathbf{u}])qr+J_{12}[\mathbf{u}]rp+\\
&\quad J_{11}^{\cdot}[\mathbf{u}]p-J_{12}^{\cdot}[\mathbf{u}]q=0\\
&J_{22}[\mathbf{u}]q^{\cdot}-J_{12}[\mathbf{u}]p^{\cdot}+(J_{11}[\mathbf{u}]-J_{33}[\mathbf{u}])pr-J_{12}[\mathbf{u}]rq-J_{12}^{\cdot}[\mathbf{u}]p+\\
&\quad J_{22}^{\cdot}[\mathbf{u}]q=0\\
&J_{33}[\mathbf{u}]r^{\cdot}+(J_{22}[\mathbf{u}]-J_{11}[\mathbf{u}])pq-J_{12}[\mathbf{u}](p^2-q^2)+\\
&\quad J_{33}^{\cdot}[\mathbf{u}]r+\int_{\Omega}[(\mathbf{r}+\mathbf{u})\times\mathbf{u}^{\cdot\cdot}]\mathbf{e}_3\rho\,dx=0
\end{aligned} \tag{13}$$

Where p, q, r are projections of vector ω on the axes $Ox_1x_2x_3$, $J_{ij}[\mathbf{u}]$ are components of matrix of inertia of system relative to axes $Ox_1x_2x_3$. In conformity with (12) the quantities $J_{ij}[\mathbf{u}]$ contain, besides constant terms sinuses and cosinuses $2n\varphi$, $n = 1, 2, 3, 4$. Moreover

$$p = P\cos\varphi + Q\sin\varphi,\quad q = -P\sin\varphi + Q\cos\varphi,\quad r = R + \varphi^{\cdot}$$
$$p^{\cdot} = (P^{\cdot} + Q\varphi^{\cdot})\cos\varphi + (Q^{\cdot} - P\varphi^{\cdot})\sin\varphi,\quad q^{\cdot} = (Q^{\cdot} - P\varphi^{\cdot})\cos\varphi - (P^{\cdot} + Q\varphi^{\cdot})\sin\varphi$$

where P, Q, R are projections of angular velocity of the system of coordinates $Oxyz$. From this follows that the first two of Eqs. (13), when averaged with respect to angle φ, become identically zero. When averaging the quantities $Q, P, r, P^{\cdot}, Q^{\cdot}, r^{\cdot}$ are assumed constants. Since $|\mathbf{u}| \ll |\mathbf{r}|$ and the derivative $r^{\cdot}$ is small, we take the averaged value of $J_{33}[\mathbf{u}]$ equal to $J_{33}[0] = C_1 + {}^1/_2 ml^2$, where C_1 is the moment of inertia of the solid body relative to the axis Ox_3, m is the mass of disk, and l its radius. The two last terms of the third of Eqs. (13) vanish. Since $|P| \ll |Q|$, then, assuming $p = Q\sin\varphi$, $q = Q\cos\varphi$, we obtain

$$\langle (J_{22}[\mathbf{u}] - J_{11}[\mathbf{u}])^{\cdot}\ {}^1/_2 Q^2 \sin 2\varphi + J_{12}[\mathbf{u}]\, Q^2 \cos 2\varphi\rangle \approx {}^1/_2 Q^2\, \Phi \tag{14}$$
$$\Phi = \int_{\Omega} [x_1(v_{12} + v_{21}) + x_2(v_{11} - v_{22})]\,\rho\, dx,\quad \langle\cdot\rangle = \frac{1}{2\pi}\int_0^{2\pi} (\cdot)\, d\varphi;$$
$$v_{ij} = \mathbf{v}_i \mathbf{e}_j,\quad i, j = 1, 2$$

where $\langle\ \rangle$ denotes averaging operation over angle φ.

The approximate equality in (14) indicates that in its right-hand side are omitted quadratic terms of vector components of displacements. As the result of computation of integral Φ we obtain

$$\Phi = (5/12)\,\chi\varphi^{\cdot} Q^2 m l^4 \mu_2 \mu_1^{-1}$$

After averaging the third of Eqs. (13), we obtain the approximate equation

$$Cr^{\cdot} + k\varphi^{\cdot} Q^4 = 0,\quad C = J_{33}[0],\quad k = 5/24\,\chi\, ml^4 \mu_2\mu_1^{-1} \tag{15}$$

We write the moment of momentum (2) in the form

$$(J_{11}[\mathbf{u}]\,p - J_{12}[\mathbf{u}]\,q)^2 + (J_{22}[\mathbf{u}]\,q - J_{12}[\mathbf{u}]\,p)^2 + \left\{J_{33}[\mathbf{u}]\,r + \int_{\Omega} [(\mathbf{r} + \mathbf{u}) \times \mathbf{u}^{\cdot}]\,\mathbf{e}_3 \rho\, dx\right\}^2 = G^2 \tag{16}$$

Averaging over the angle φ and limiting to the principal terms (actually it is necessary to set in (16) $\mathbf{u} \equiv 0$)), we obtain

$$A^2Q^2 + C^2r^2 = G^2,\quad A = A_1 + {}^1/_4\, ml^2 \tag{17}$$

where A_1 is the solid body moment of inertia relative to the axis Ox_1. We use the relation $\varphi^{\cdot} = r(A - C)/A$, which is valid in the case of regular precession in the absence of extraneous forces, and from Eqs. (15) and (17) obtain

$$\gamma^{\cdot} = n\gamma(1 - \gamma^2)^2,\quad n = \frac{kG^4(G - A)}{CA^5},\quad \gamma = \frac{Cr}{G} = \cos\theta \tag{18}$$

Equation (18) defines evolution of angle θ. If $A < C$ then $n > 0$ and $\lim \gamma(t) = \pm 1$ as $t \to \infty$; depending on weather $\gamma(0)$ is greater or smaller than zero. With this angle θ approaches zero or π, i.e. the body tends to rotate about the axis of symmetry. For $A > C$ we have $n < 0$ and $\lim \gamma(t) = 0$ for $t \to \infty$. This means that $\lim \theta(t) = {}^1/_2\pi$ and $t \to \infty$, the body approaches steady rotation about one of the diameters belonging to the equatorial plane of the ellipsoid of inertia.

Equation (18) has three steady solutions $\gamma = 0$ and $\gamma = \pm 1$. The solution $\gamma = 0$ $(\theta = {}^1/_2\pi)$ is stable for $A > C$ and unstable for $A < C$, while solution $\gamma = \pm 1$ $(\theta = 0, \pi)$ is stable for $A < C$ and unstable for $A > C$.

The angular velocity of precession $\psi^{\cdot} = GA^{-1}$ remains constant in the evolution of motion, and $\gamma(t)$ is given implicitly by the relation $\ln[(1 - \gamma^2)\gamma^{-2}] - \gamma^2(1 - \gamma^2)^{-1} = -2nt + \ln[(1 - \gamma^2(0))\gamma^{-2}(0)] - \gamma^2(0)(1 - \gamma^2)(0))^{-1}$.

REFERENCES

1. APPEL P., Theoretical Mechanics, Traite de Mécanique Nationale Vol.2., Dynamique des Sŷstèmes, 1893.
2. VIL'KE V.G., The Limit motion of a system of heavy elastic solid body with a fixed point. Vestn. MGU Ser. Matem, Mekhan., No.3, 1980.

3. LEIBENZON L.S., Short Course of the Theory of Elasticity. Moscow – Leningrad, GOSTEKHIZDAT, 1942.
4. CHERNOUS'KO F.L., On the motion of a solid body with elastic and dissipative elements. PMM, Vol.42, No.1, 1978.
5. VIL'KE V.G., Motion of a visco-elastic sphere in a central Newtonian force field. PMM, Vol.44, No.3, 1980.

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